Percolation critical exponents in scale-free networks

Reuven Cohen,[1,*] Daniel ben-Avraham,[2] and Shlomo Havlin[1]

[1]*Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan, Israel*
[2]*Department of Physics, Clarkson University, Potsdam, New York 13699-5820*
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We study the behavior of scale-free networks, having connectivity distribution $P(k) \sim k^{-\lambda}$, close to the percolation threshold. We show that for networks with $3 < \lambda < 4$, known to undergo a transition at a finite threshold of dilution, the critical exponents are different than the expected mean-field values of regular percolation in infinite dimensions. Networks with $2 < \lambda < 3$ possess only a percolative phase. Nevertheless, we show that in this case percolation critical exponents are well defined, near the limit of extreme dilution (where all sites are removed), and that also then the exponents bear a strong λ dependence. The regular mean-field values are recovered only for $\lambda > 4$.

Large networks have been attracting considerable interest [1–6]. A useful characterization of such networks is their degree distribution $P(k)$ or the probability that an arbitrary node be connected to exactly k other nodes. Many naturally occurring networks (the Internet, nets of social contacts, scientific collaborations, ecological nets of predator-prey, etc.,) exhibit a power-law or *scale-free* degree distribution:

$$P(k) = ck^{-\lambda}, \quad k \geq m, \tag{1}$$

where m is the minimal connectivity (usually taken to be $m = 1$) and c is a normalization factor.

Imagine that a large network is diluted, by random removal of a fraction p of its nodes. If p is small, one expects that the network remains essentially connected: there exists a *giant component* of connected nodes that constitutes a finite fraction of the total size of the original network. Above a certain threshold of dilution p_c, the giant component disappears and the network effectively disintegrates. This is the problem of percolation [7,8] as defined on networks. (Percolation was originally defined on regular lattices.) The giant component corresponds to the *infinite incipient cluster* or *spanning cluster* which forms only in the *percolating phase*, at dilutions smaller than the percolation threshold p_c.

Percolation in scale-free networks is widely recognized as a key problem of interest [1–6]. Applications range from the robustness of communication networks (the Internet, phone networks) in the face of random failure (removal) of a fraction of their nodes (e.g., random breakdown of routers), to the efficiency of inoculation strategies against the spread of disease, to the ecological impact of the extinction of some species.

For a random network of arbitrary degree distribution, the condition for the existence of a spanning cluster is [5,9–11]

$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} > 2. \tag{2}$$

*Email address: cohenr@shoshi.ph.biu.ac.il

Suppose a fraction p of the nodes (and their links) are removed from the network. (Alternatively, a fraction $q = 1 - p$ of the nodes are retained.) The original degree distribution, $P(k)$, becomes

$$P'(k) = \sum_{l=k}^{\infty} P(l) \binom{l}{k} (1-p)^k p^{l-k}. \tag{3}$$

In view of this new $P'(k)$, Eq. (2) yields the percolation threshold:

$$q_c = 1 - p_c = \frac{1}{\kappa - 1}, \tag{4}$$

where κ is computed with respect to the original distribution $P(k)$ before dilution.

Applying the criterion (4) to scale-free networks one concludes that for $\lambda > 3$ a phase transition exists at a finite q_c, whereas for $2 < \lambda < 3$ the transition takes place only at the extreme limit of dilution of $q_c = 0$ [5,6]. Here we concern ourselves with the critical exponents associated with the percolation transition in scale-free networks of $\lambda > 2$. We show that critical exponents are well defined (and we compute their λ-dependent values) even in the nonpercolative regime of $2 < \lambda < 3$. In the percolative range of $\lambda > 3$, we find λ-dependent exponents which differ from the regular mean-field values of percolation in infinite dimensions. This irregularity can be traced to the fat tail of the distribution (1). The regular mean-field exponents are recovered only for $\lambda > 4$.

In [4,12] a generating function is built for the connectivity distribution:

$$G_0(x) = \sum_{k=0}^{\infty} P(k) x^k. \tag{5}$$

The probability of reaching a site with connectivity k by following a specific link is $kP(k)/\langle k \rangle$ [4,9,12], and the corresponding generating function for those probabilities is

$$G_1(x) = \frac{\sum kP(k)x^{k-1}}{\sum kP(k)} = \frac{d}{dx}G_0(x)/\langle k \rangle. \quad (6)$$

Let $H_1(x)$ be the generating function for the probability of reaching a branch of a given size by following a link. After a dilution of a fraction p of the sites, $H_1(x)$ satisfies the self-consistent equation

$$H_1(x) = 1 - q + qxG_1(H_1(x)). \quad (7)$$

Since $G_0(x)$ is the generating function for the connectivity of a site, the generating function for the probability of a site to belong to an n-site cluster is

$$H_0(x) = 1 - q + qxG_0(H_1(x)). \quad (8)$$

$H_0(1)$ is the probability that a site belongs to a cluster of any *finite* size. Thus, below the percolation transition $H_0(1) = 1$, while above the transition there is a finite probability that a site belongs to the infinite spanning cluster: $P_\infty = 1 - H_0(1)$. It follows that

$$P_\infty(q) = q\left(1 - \sum_{k=0}^\infty P(k)u^k\right), \quad (9)$$

where $u \equiv H_1(1)$ is the smallest positive root of

$$u = 1 - q + \frac{q}{\langle k \rangle}\sum_{k=0}^\infty kP(k)u^{k-1}. \quad (10)$$

This equation can be solved numerically and the solution may be substituted into Eq. (9), yielding the size of the spanning cluster in a network of arbitrary degree distribution, at dilution q [4].

We now compute the order parameter critical exponent β. Near criticality the probability of belonging to the spanning cluster behaves as $P_\infty \sim (q - q_c)^\beta$. For infinite-dimensional systems (such as a Cayley tree) it is known that $\beta = 1$ [7,8,13]. This regular mean-field result is not always valid, however, for scale-free networks. Eq. (9) has no special behavior at $q = q_c$; the singular behavior comes from u. Also, at criticality $P_\infty = 0$ and Eq. (9) implies that $u = 1$. We therefore examine Eq. (10) for $u = 1 - \epsilon$ and $q = q_c + \delta$:

$$1 - \epsilon = 1 - q_c - \delta + \frac{(q_c + \delta)}{\langle k \rangle}\sum_{k=0}^\infty kP(k)(1 - \epsilon)^{k-1}. \quad (11)$$

The sum in Eq. (11) has the asymptotic form

$$\sum_{k=0}^\infty kP(k)u^{k-1} \sim \langle k \rangle - \langle k(k-1) \rangle \epsilon + \frac{1}{2}\langle k(k-1)(k-2) \rangle \epsilon^2$$

$$+ \cdots + c\Gamma(2-\lambda)\epsilon^{\lambda-2}, \quad (12)$$

where the highest-order analytic term is $O(\epsilon^n)$, $n = \lfloor \lambda - 2 \rfloor$. Using this in Eq. (11), with $q_c = 1/(\kappa - 1) = \langle k \rangle/\langle k(k-1) \rangle$, we get

$$\frac{\langle k(k-1) \rangle^2}{\langle k \rangle}\delta = \frac{1}{2}\langle k(k-1)(k-2) \rangle\epsilon + \cdots + c\Gamma(2-\lambda)\epsilon^{\lambda-3}. \quad (13)$$

The divergence of δ as $\lambda < 3$ confirms the lack of a phase transition in that regime. Thus, limiting ourselves to $\lambda > 3$, and keeping only the dominant term as $\epsilon \to 0$, Eq. (13) implies

$$\epsilon \sim \begin{cases} \left(\dfrac{\langle k(k-1) \rangle^2}{c\langle k \rangle\Gamma(2-\lambda)}\right)^{(1/\lambda-3)}\delta^{(1/\lambda-3)}, & 3 < \lambda < 4 \\[3mm] \dfrac{2\langle k(k-1) \rangle^2}{\langle k \rangle\langle k(k-1)(k-2) \rangle}\delta, & \lambda > 4. \end{cases} \quad (14)$$

Returning to P_∞, Eq. (9), we see that the singular contribution in ϵ is dominant only for the irrelevant range of $\lambda < 2$. For $\lambda > 3$, we find $P_\infty \sim q_c\langle k \rangle\epsilon \sim (q - q_c)^\beta$. Comparing this to Eq. (14) we finally obtain

$$\beta = \begin{cases} \dfrac{1}{\lambda - 3}, & 3 < \lambda < 4, \\[3mm] 1, & \lambda > 4. \end{cases} \quad (15)$$

We see that the order parameter exponent β attains its usual mean-field value only for $\lambda > 4$. Moreover, for $\lambda < 4$ the percolation transition is higher than second-order: for $3 + [1/(n-1)] < \lambda < 3 + [1/(n-2)]$ the transition is of the nth order. The result (15) has been reported before in [6], and also found independently in a different but related model of virus spreading [14,15]. The existence of an infinite-order phase transition at $\lambda = 3$ for growing networks of the Albert-Barabási model, has been reported elsewhere [16,17]. These examples suggest that the critical exponents are not model dependent but depend only on λ.

For networks with $\lambda < 3$ the transition still exists, though at a vanishing threshold, $q_c = 0$. The sum in Eq. (11) becomes

$$\sum_{k=0}^\infty kP(k)u^{k-1} \sim \langle k \rangle + c\Gamma(2-\lambda)\epsilon^{\lambda-2}. \quad (16)$$

Using this in conjunction with Eq. (10), and remembering that here $q_c = 0$ and therefore $q = \delta$, leads to

$$\epsilon = \left(\frac{-c\Gamma(2-\lambda)}{\langle k \rangle}\right)^{(1/3-\lambda)}\delta^{(1/3-\lambda)}, \quad (17)$$

which implies

$$\beta = \frac{1}{3 - \lambda}, \quad 2 < \lambda < 3. \quad (18)$$

In other words, the transition in $2 < \lambda < 3$ is a mirror image of the transition in $3 < \lambda < 4$. An important difference is that $q_c = 0$ is not λ dependent in $2 < \lambda < 3$, and the amplitude of P_∞ diverges as $\lambda \to 2$ (but remains finite as $\lambda \to 4$).

In [12] it was shown that for a random graph of arbitrary degree distribution the finite clusters follow the usual scaling form

$$n_s \sim s^{-\tau} e^{-s/s*}. \tag{19}$$

Here s is the cluster size and n_s is the number of clusters of size s. At criticality $s* \sim |q - q_c|^{-\sigma}$ diverges and the tail of the distribution behaves as a power law. We now derive the exponent τ. The probability that a site belongs to an s cluster is $p_s = s n_s \sim s^{1-\tau}$, and is generated by H_0:

$$H_0(x) = \sum p_s x^s. \tag{20}$$

The singular behavior of $H_0(x)$ stems from $H_1(x)$, as can be seen from Eq. (8). $H_1(x)$ itself can be expanded from Eq. (7), by using the asymptotic form (12) of G_1. We let $x = 1 - \epsilon$, as before, but work at the critical point $q = q_c$. With the notation $\phi(\epsilon) = 1 - H_1(1 - \epsilon)$, we finally get [note that at criticality $H_1(1) = 1$]

$$-\phi = -q_c + (1-\epsilon)q_c \left[1 - \frac{\phi}{q_c} + \frac{\langle k(k-1)(k-2) \rangle}{2 \langle k \rangle} \phi^2 + \cdots \right.$$

$$\left. + c \frac{\Gamma(2-\lambda)}{\langle k \rangle} \phi^{\lambda-2} \right]. \tag{21}$$

From this relation we extract the singular behavior of H_0: $\phi \sim \epsilon^y$. Then, using Tauberian theorems [18] it follows that $p_s \sim s^{-1-y}$, hence $\tau = 2 + y$.

For $\lambda > 4$ the term proportional to $\phi^{\lambda-2}$ in Eq. (21) may be neglected. The linear term $\epsilon \phi$ may be neglected as well, due to the factor ϵ. This leads to $\phi \sim \epsilon^{1/2}$ and to the usual mean-field result

$$\tau = \frac{5}{2}, \quad \lambda > 4. \tag{22}$$

For $\lambda < 4$, the terms proportional to $\epsilon \phi$, ϕ^2 may be neglected, leading to $\phi \sim \epsilon^{1/(\lambda-2)}$ and

$$\tau = 2 + \frac{1}{\lambda-2} = \frac{2\lambda-3}{\lambda-2}, \quad 2 < \lambda < 4. \tag{23}$$

Note that for $2 < \lambda < 3$ the percolation threshold is strictly $q_c = 0$. In that case we work at $q = \delta$, small but fixed, taking the limit $\delta \to 0$ at the very end [19]. For growing networks of the Albert-Barabási model with $\lambda = 3$, it has been shown that $s n_s \propto (s \ln s)^{-2}$ [17]. This is consistent with $\tau = 3$ plus a logarithmic correction. Related results for scale-free trees have been presented in [20].

The critical exponent σ, for the cutoff cluster size, too may be derived directly. Finite-size scaling arguments predict [7] that

$$q_c(\infty) - q_c(N) \sim N^{-1/d\nu} = N^{-(\sigma/\tau-1)}, \tag{24}$$

where N is the number of sites in the network, ν is the correlation length critical exponent: $\xi \sim (q - q_c)^{-\nu}$, and d is the dimensionality of the embedding space. Using a continuous approximation of the distribution (1) one obtains [5]

$$\kappa \approx \left(\frac{2-\lambda}{3-\lambda} \right) \frac{K^{3-\lambda} - m^{3-\lambda}}{K^{2-\lambda} - m^{2-\lambda}}, \tag{25}$$

where $K \sim N^{1/(\lambda-1)}$ is the largest site connectivity of the network. For $3 < \lambda < 4$, this and Eq. (4) yield

$$q_c(\infty) - q_c(N) \sim \Delta \kappa \sim K^{3-\lambda} \sim N^{(3-\lambda/\lambda-1)}, \tag{26}$$

which in conjunction with Eq. (24) leads to

$$\sigma = \frac{\lambda-3}{\lambda-2}, \quad 3 < \lambda < 4. \tag{27}$$

For $\lambda > 4$ we recover the regular mean-field result $\sigma = 1/2$. Note that Eqs. (24), (15), and (23) are consistent with the known scaling relation: $\sigma \beta = \tau - 2$ [7,8,13]. For $2 < \lambda < 3$, $q_c(\infty) = 0$ and $q_c(N) \sim K^{\lambda-3} \sim N^{(\lambda-3)/(\lambda-1)}$. Therefore,

$$\sigma = \frac{3-\lambda}{\lambda-2}, \quad 2 < \lambda < 3, \tag{28}$$

again consistent with the scaling relation $\sigma \beta = \tau - 2$ [cf. Eq. (18)].

Through similar scaling relations, any two of the percolation exponents discussed above determine the remaining (static) percolation exponents. Thus, for example, the exponent γ, governing the average size of finite clusters: $\langle s \rangle \sim (q - q_c)^{-\gamma}$, follows from the scaling relation $\gamma = (3 - \tau)/\sigma$. In view of Eqs. (23), (27), and (28), we get

$$\gamma = \begin{cases} 1, & \lambda > 3 \\ -1, & 2 < \lambda < 3. \end{cases} \tag{29}$$

The negative value of γ in the range $2 < \lambda < 3$ is, at first sight, surprising. However, recall that for that range only the percolating phase exists. At the transition point, $q_c = 0$, all clusters have size *zero*. As q increases, finite-size clusters begin to show, consistent with $\langle s \rangle \sim (q - q_c)^{-\gamma} \sim q$.

In summary, we have shown that percolation critical exponents in scale-free networks bear a strong dependence upon the degree distribution exponent λ. This is true even in the range $3 < \lambda < 4$, where the percolation transition occurs at a finite threshold q_c: the regular mean-field behavior of percolation in infinite dimensions is recovered only for $\lambda > 4$. Moreover, critical exponents are well defined also in the most physically relevant region of $2 < \lambda < 3$ (the case of most naturally occurring networks), despite the lack of a nonpercolating phase. In this regime too the critical exponents depend strongly on λ.

An interesting exception is the exponent γ, which attains constant values independent of λ, Eq. (29). This suggests some underlying deeper principle which we were not yet able to uncover.

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